(2) Power series representation. Lecture 2

Def. 2 A series $\sum_{\alpha \in A} a_\alpha(z)$ converges **normally** in Ω if $\sum_\alpha |a_\alpha(z)|$ converges uniformly on compact subsets of Ω.

Clearly, normal convergence $\Rightarrow$ unif. conv. on compact subsets.

Notation. $D^n_{a,r}$ is PD w/ center a and polyradius r

Thm 2. Let $\overline{D^n_{a,r}} \subseteq \Omega$ and f holom. in Ω. Then for $z \in D^n$.

$$f(z) = \sum_{\alpha \in \mathbb{Z}^n_+} a_\alpha (z-a)^\alpha$$

$$(z_1-a_1)^{\alpha_1} \cdots (z_n-a_n)^{\alpha_n}$$

w/ normal convergence in D^n.

Moreover, $a_\alpha = \frac{1}{\alpha!} \frac{\partial^\alpha f}{\partial z^\alpha}(a)$,

Rem. (i) There are many different $D^n_{a,r}$ (different $r \in \mathbb{R}^n_+$) centered at $a \in \Omega$ that are maximal w/ $\overline{D^n_{a,r}} \subseteq \Omega$. Compare this to 1-D, where there is only one.

(ii) PS expansion $\Rightarrow$ uniqueness of hol. fcns.

(3) $\bar\partial$-equation in D^n

We consider $(0,1)$-forms in $\mathbb{C}^n$
$f = \sum_{j=1}^{n} f_j \, d\bar z_j$, where $f_j \in C^k$ $(k \geq 1)$.

We introduce the $\bar\partial$-operator on functions u:

$$\bar\partial u = \sum_{j=1}^{n} \frac{\partial u}{\partial \bar z_j} d\bar z_j.$$

Thus, $\bar\partial u$ is a $(0,1)$-form.
An important tool in SCV is to solve $\bar\partial$-equation: Given $(0,1)$-form f, solve the equation

$$\bar\partial u = f.$$

We note that this is an overdetermined system of PDE

$$\frac{\partial u}{\partial \bar z_j} = f_j, \quad j = 1, \dots, n.$$

If we expect a $\mathcal{C}^k$ solution $(k \geq 2)$ we see that we must have the compatibility conditions

$$\left(\frac{\partial^2 u}{\partial \bar{z}_j \partial \bar{z}_k} = \right) \frac{\partial f_j}{\partial \bar{z}_k} = \frac{\partial f_k}{\partial \bar{z}_j} \left(= \frac{\partial^2 u}{\partial \bar{z}_k \partial \bar{z}_j} \right)$$

If we introduce the action of $\bar{\partial}$ on $(0,1)$-forms

$$\bar{\partial} f = \sum_{j,k} \frac{\partial f_j}{\partial \bar{z}_k} d\bar{z}_k \wedge d\bar{z}_j ,$$

yielding a $(0,2)$-form, then if we write $\bar{\partial} f = \sum_{j<k} \left(\frac{\partial f_k}{\partial \bar{z}_j} - \frac{\partial f_j}{\partial \bar{z}_k} \right) d\bar{z}_j \wedge d\bar{z}_k$

we can phrase the compatibility condition $\bar{\partial} f = 0$. Another way of expressing the necessity of this condition is that $\bar{\partial}^2 = \bar{\partial} \circ \bar{\partial} = 0$.

Nevertheless, it is not trivial to solve the $\bar\partial$-equation and it is only solvable in certain situations, as we shall discuss. But there is a simple situation where we can solve $\bar\partial$ easily.

<u>Thm 3.</u> Let $f = \sum f_j \, d\bar z_j$ be a $(0,1)$ form in $\mathbb{C}^n$ w/ compact support. $\mathcal{C}^k \ (k \geq 1)$
If $\bar\partial f = 0$ and $\underline{\underline{n > 1}}$, then $\exists\, u \in \mathcal{C}^k$ w/ compact support s.t. $\bar\partial u = f$.

Let's first discuss the case $n=1$. In this case, the condition $\bar\partial f = 0$ is vacuous. Nevertheless, you can solve $\bar\partial u = f$, but <u>not w/ compact support</u>.

To show how we can solve $\bar\partial u = f$ when $f = f_1 \, d\bar z$ and f_1 has compact support in, say, $K \subset\subset \mathbb{C}$,